Convolution

$$k * f(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} k(x,y)\, f(\alpha - x, \beta - y)$$

$$= \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} k(x,y)\, f(\alpha - x, \beta - y)$$

Reverse the order

$$= \begin{array}{l} k(0,0) f(\alpha,\beta) + \boxed{k(0,1) f(\alpha, \beta-1) + \cdots + k(0,N-1) f(\alpha, \beta-N+1)} \\ + \; k(1,0) f(\alpha-1, \beta) + k(1,1) f(\alpha-1, \beta-1) + \cdots + k(1,N-1) f(\alpha-1, \beta-N+1) \\ + \quad \vdots \qquad\qquad\qquad\qquad \ddots \qquad\qquad\qquad \vdots \\ + \; k(N-1,0) f(\alpha-N+1, \beta) + k(N-1,1) f(\alpha-N+1, \beta-1) + \cdots + k(N-1,N-1) f(\alpha-N+1, \beta-N+1) \end{array}$$

Reverse the order

Rotate 180°

$$= \begin{array}{l} k(0,0) f(\alpha,\beta) + k(0,N-1) f(\alpha, \beta-N+1) + \cdots + k(0,1) f(\alpha, \beta-1) \\ + \; k(N-1,0) f(\alpha-N+1, \beta) + k(N-1,N-1) f(\alpha-N+1, \beta-N+1) + \cdots + k(N-1,1) f(\alpha-N+1, \beta-1) \\ + \quad \vdots \qquad\qquad \vdots \qquad\qquad \ddots \qquad\qquad \vdots \\ + \; k(1,0) f(\alpha-1, \beta) + k(1,N-1) f(\alpha-1, \beta-N+1) + \cdots + k(1,1) f(\alpha-1, \beta-1) \end{array}$$

Periodic :

$$= \begin{array}{l} k(0,0) f(\alpha,\beta) + k(0,N-1) f(\alpha, \beta+1) + \cdots + k(0,1) f(\alpha, \beta+N-1) \\ + \; k(N-1,0) f(\alpha+1, \beta) + k(N-1,N-1) f(\alpha+1, \beta+1) + \cdots + k(N-1,1) f(\alpha+1, \beta+N-1) \\ + \quad \vdots \qquad\qquad \vdots \qquad\qquad \ddots \qquad\qquad \vdots \\ + \; k(1,0) f(\alpha+N-1, \beta) + k(1,N-1) f(\alpha+N-1, \beta+1) + \cdots + k(1,1) f(\alpha+N-1, \beta+N-1) \end{array}$$

$$k \quad \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ k_{21} & k_{22} & \cdots & k_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N1} & k_{N2} & \cdots & k_{NN} \end{bmatrix} \xrightarrow{\text{Rotate}} \begin{bmatrix} k_{NN} & \cdots & k_{N2} & k_{N1} \\ \vdots & \ddots & \vdots & \vdots \\ k_{2N} & \cdots & k_{22} & k_{21} \\ k_{1N} & \cdots & k_{12} & k_{11} \end{bmatrix}$$

put on $\downarrow$

$$\begin{bmatrix} f_{\alpha\beta} & f_{\alpha,\beta+1} & \cdots & f_{\alpha,\beta+N-1} \\ f_{\alpha+1,\beta} & f_{\alpha+1,\beta+1} & \cdots & f_{\alpha+1,\beta+N-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{\alpha+N-1,\beta} & f_{\alpha+N-1,\beta+1} & \cdots & f_{\alpha+N-1,\beta+N-1} \end{bmatrix}$$

In fact the same

$$k \quad \begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0,N-1} \\ k_{10} & k_{11} & \cdots & k_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N-1,0} & k_{N-1,1} & \cdots & k_{N-1,N-1} \end{bmatrix} \longrightarrow \begin{bmatrix} k_{00} & k_{0,N-1} & \cdots & k_{0,1} \\ k_{N-1,0} & k_{N-1,N-1} & \cdots & k_{N-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,0} & k_{1,N-1} & \cdots & k_{1,1} \end{bmatrix}$$

put on $\downarrow$

$$\begin{bmatrix} f_{\alpha\beta} & f_{\alpha,\beta+1} & \cdots & f_{\alpha,\beta+N-1} \\ f_{\alpha+1,\beta} & f_{\alpha+1,\beta+1} & \cdots & f_{\alpha+1,\beta+N-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{\alpha+N-1,\beta} & f_{\alpha+N-1,\beta+1} & \cdots & f_{\alpha+N-1,\beta+N-1} \end{bmatrix}$$

Example

$$k * f(\alpha,\beta) := \overset{k(0,0)}{-4f(\alpha,\beta)} + \overset{k(-1,0)}{f(\alpha+1,\beta)} + \overset{k(1,0)}{f(\alpha-1,\beta)} + \overset{k(0,-1)}{f(\alpha,\beta+1)} + \overset{k(0,1)}{f(\alpha,\beta-1)}$$

$$k(0,0) = -4$$

$$k(-1,0) = 1 \qquad k(1,0) = 1$$

$$k(0,-1) = 1 \qquad k(0,1) = 1$$

In Indexing starting from 1:

$$k = \begin{bmatrix} & & & & & 1 \\ & & & & & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ & & & & & 1 \\ 1 & 0 & \cdots & 0 & 1 & -4 \end{bmatrix}$$

In Indexing starting from 0:

$$k = \begin{bmatrix} -4 & 1 & 0 & \cdots & 0 & 1 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \\ 1 & & & & & \end{bmatrix}$$

Image Sharpening

$f \in \mathbb{R}^{N \times N}$ clean image

Δf is Discrete Laplacian of f

(in continuous setting,

$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ if $f : \mathbb{R}^2 \to \mathbb{R}$)

One discretization of Δ is given by:

$$
\begin{aligned}
\Delta f(x,y) = {} & \underbrace{\underbrace{[f(x+1,y) - f(x,y)]}_{\approx \frac{\partial f}{\partial x}} - \underbrace{[f(x,y) - f(x-1,y)]}_{\approx \frac{\partial f}{\partial x}}}_{\approx \frac{\partial^2 f}{\partial x^2}} \\
& + \underbrace{\underbrace{[f(x,y+1) - f(x,y)]}_{\approx \frac{\partial f}{\partial y}} - \underbrace{[f(x,y) - f(x,y-1)]}_{\approx \frac{\partial f}{\partial y}}}_{\approx \frac{\partial^2 f}{\partial y^2}} \\
= {} & -4 f(x,y) + f(x+1,y) + f(x-1,y) \\
& + f(x,y+1) + f(x,y-1)
\end{aligned}
$$

Δf can be understended as measuring the smoothness of f.

When there is a Peak:

$\frac{\partial^2 f}{\partial x^2} < 0$ and $\frac{\partial^2 f}{\partial y^2} < 0$

$\Rightarrow$ Δf "very negative"

When there is a valley:

$\frac{\partial^2 f}{\partial x^2} > 0$, $\frac{\partial^2 f}{\partial y^2} > 0$

$\Rightarrow$ Δf "very positive"

$f - \Delta f$ is unsmoothing f,

"peak become more peak"

"valley become more valley"

This is Called Laplacian Masking.

$\tilde{f} \in \mathbb{R}^{N \times N}$ smoothed f.

$\tilde{f}$ is smoothed, means that $\tilde{f}$ has less details.

$f - \tilde{f}$ is the removed details.

$$\underbrace{g}_{\text{sharpened}} = f + k \underbrace{(f - \tilde{f})}_{\text{add more details to the original image}}$$

Unsharp Masking: $k = 1$

e.g. If f is smoothed by a low pass filter to $\tilde{f}$,

Then $$DFT(\tilde{f})(u,v) = \underbrace{H(u,v)}_{\text{low pass filter.}} DFT(f)(u,v)$$

$$DFT(g)(u,v) = (2 - H(u,v))\, DFT(f)(u,v)$$

If Butterworth-low-pass filter is applied,

$$DFT(g)(u,v) = \frac{1 + 2(D(u,v)/D_0)^n}{1 + (D(u,v)/D_0)^n} DFT(f)(u,v)$$

PDE approach (See also Heat / Diffusion Equation)

Consider

space   time

$$\frac{\partial I(x,y;t)}{\partial t} = t\left[\frac{\partial^2 I(x,y;t)}{\partial x^2} + \frac{\partial^2 I(x,y;t)}{\partial y^2}\right]$$

where $(x,y) \in \mathbb{R}^2$, $t > 0$,

$I(x,y;t) \in \mathbb{R}$

You can understand I as a movie, where the next frame is obtained by modifying the previous frame.

Let $I_0(u,v)$ be an image. (first frame)

$$g(x,y;t) := \frac{1}{2\pi t^2} e^{-(x^2+y^2)/2t^2}$$

First Note that g satisfy the PDE:

$$\frac{\partial g}{\partial t} = -\frac{1}{\pi t^3} e^{-(x^2+y^2)/2t^2} + \frac{1}{2\pi t^2} e^{-(x^2+y^2)/2t^2}\left(\frac{x^2+y^2}{t^3}\right)$$

$$\frac{\partial g}{\partial x} = -\frac{1}{2\pi t^2} e^{-(x^2+y^2)/2t^2}\left(\frac{x}{t^2}\right)$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{1}{2\pi t^2} e^{-(x^2+y^2)/2t^2}\left(\frac{x}{t^2}\right)^2 - \frac{1}{2\pi t^2} e^{-(x^2+y^2)/2t^2}\left(\frac{1}{t^2}\right)$$

Similar for $\frac{\partial^2 g}{\partial y^2}$,

then $\frac{\partial g}{\partial t} = t \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right)$.

continuous 2D convolution

$$I(x, y; t) := g * I_0(x, y)$$

$$= \int_{\mathbb{R}^2} g(u, v; t)\, I_0(x-u, y-v)\, du\, dv$$

derivative on time, not related to space

integration on space, not related to time

$$\frac{\partial I}{\partial t} = \frac{\partial}{\partial t} \int_{\mathbb{R}^2} g(u, v; t)\, I_0(x-u, y-v)\, du\, dv$$

(If g, $\frac{\partial g}{\partial t}$ are continuous, we can interchange $\int_{\mathbb{R}^2}$ and $\frac{\partial}{\partial t}$. In the course, you may assume it holds)

$$\frac{\partial I}{\partial t}(x, y; t) = \int_{\mathbb{R}^2} \frac{\partial g}{\partial t}(u, v; t)\, I_0(x-u, y-v)\, du\, dv$$

$$= \int_{\mathbb{R}^2} t \frac{\partial^2 g}{\partial u^2}\, I_0(x-u, y-v)\, du\, dv$$

$$+ \int_{\mathbb{R}^2} t \frac{\partial^2 g}{\partial v^2}\, I_0(x-u, y-v)\, du\, dv$$

Integration by Part, (g, I_0 has to be vanished at $\infty, -\infty$ such that the improper integration make sense)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t\, g(u,v;t) \frac{\partial^2}{\partial u^2} I_0(x-u, y-v)\, du\, dv$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t\, g(u,v;t) \frac{\partial^2}{\partial v^2} I_0(x-u, y-v)\, du\, dv$$

Change of variable: $x' = x-u$, $y' = y-v$,

$$= \int_{\infty}^{-\infty}\int_{\infty}^{-\infty} t\, g(x-x', y-y'; t) \left[(-1)^2 \frac{\partial^2}{\partial x'^2} I_0(x', y')\right](-dx')(-dy')$$

$$+ \int_{\infty}^{-\infty}\int_{\infty}^{-\infty} t\, g(x-x', y-y'; t) \left[(-1)^2 \frac{\partial^2}{\partial y'^2} I_0(x', y')\right](-dx')(-dy')$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t\, g(x-x', y-y'; t) \frac{\partial^2}{\partial x'^2} I_0(x', y')\, dx'\, dy'$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t\, g(x-x', y-y'; t) \frac{\partial^2}{\partial y'^2} I_0(x', y')\, dx'\, dy'$$

Note $\frac{\partial}{\partial x}(f_1 * f_2) = \frac{\partial f_1}{\partial x} * f_2 = f_1 * \frac{\partial f_2}{\partial x}$

(exercise)

Then

$$\frac{\partial I}{\partial t} = t\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right) \int_{\mathbb{R}^2} g(x-x', y-y'; t)\, I_0(x', y')\, dx'\, dy'$$

$= g * I_0 = I$

$$\frac{\partial I}{\partial t} = t\, \Delta I.$$

$\therefore$ $I = g * I_0$ solve the PDE

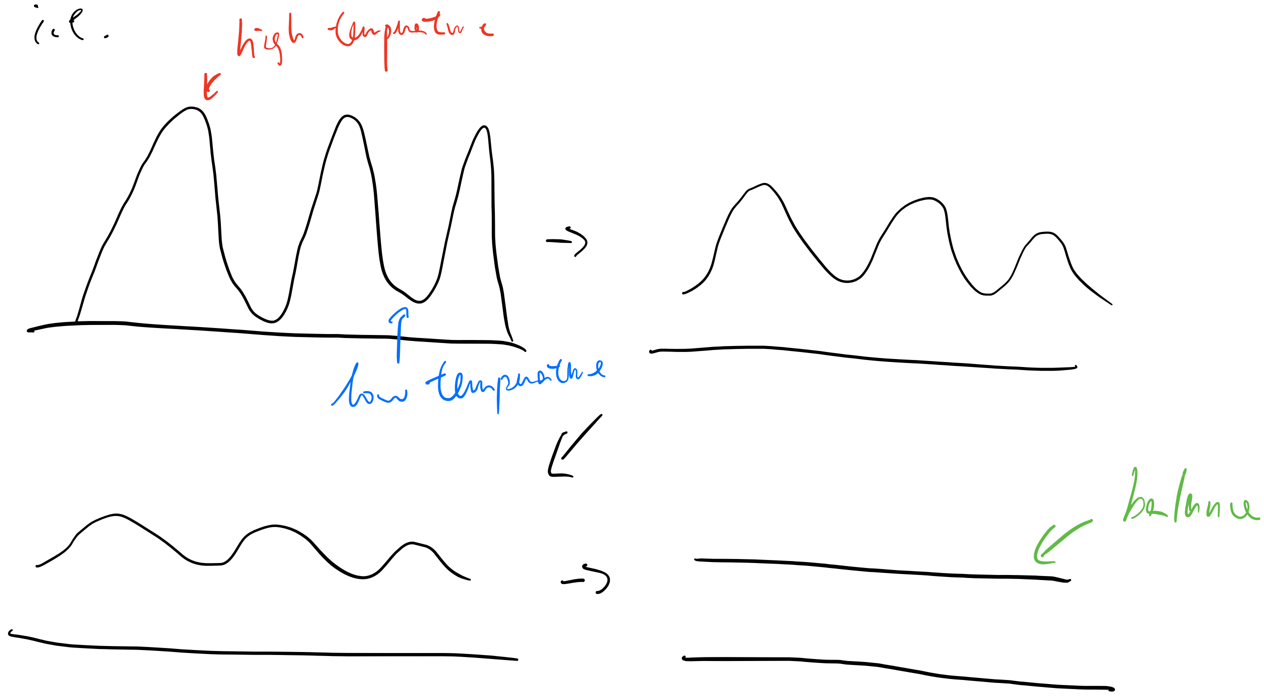
Idea for using Heat Equation,

In Physics,

the heat energy will spread out,

the temperature will tends to balance

i.e.



The distribution of heat energy is smoothing

as $t \to \infty$